General Extinction Results for Stochastic Partial Differential Equations and Applications*

Michael Röckner[b)], **Feng-Yu Wang**[a),c)†]

[a)]School of Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China

[b)]Department of Mathematics, Bielefeld University, D-33501 Bielefeld, Germany

[c)]Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK

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Abstract

Let L be a positive definite self-adjoint operator on the L^2-space associated to a σ-finite measure space. Let H be the dual space of the domain of $L^{1/2}$ w.r.t. $L^2(\mu)$. By using an Itô type inequality for the H-norm and an integrability condition for the hyperbound of the semigroup $P_t := \mathrm{e}^{-Lt}$, general extinction results are derived for a class of continuous adapted processes on H. Main applications include stochastic and deterministic fast diffusion equations with fractional Laplacians. Furthermore, we prove exponential integrability of the extinction time for all space dimensions in the singular diffusion version of the well-known Zhang-model for self-organized criticality, provided the noise is small enough. Thus we obtain that the system goes to the critical state in finite time in the deterministic and with probability one in finite time in the stochastic case.

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1 Introduction

The phenomenon of self-organized criticality (SOC) is widely studied in physics from different perspectives. In [2] it was proposed to describe this phenomenon, e.g. in the

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†Corresponding author. wangfy@bnu.edu.cn; F.Y.Wang@swansea.ac.uk

case of the avalanche dynamics in the Bak-Tang-Wiesenfeld-model (see [1]) and in the Zhang-model (see [13]) by a singular diffusion with the incoming energy being realized by adding a noise term, for instance a linear multiplicative noise. The resulting stochastic partial diffusion equation is then of the following type:

$$\mathrm{d}X_t - \Delta\Psi(X_t)\mathrm{d}t \ni B(X_t)\mathrm{d}W_t, \tag{1.1}$$

where $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued map defined by

$$\Psi(s) := \begin{cases} \theta_1 + \theta_2 s, & \text{if } s \in [0,\infty), \\ [0,\theta_1], & \text{if } s = 0, \\ 0, & \text{if } s < 0, \end{cases} \tag{1.2}$$

where $\theta_1, \theta_2 \geq 0$ are constants, with $\theta_2 = 0$ in the BTW-model whereas $\theta_2 > 0$ in the Zhang-model. Hence in the BTW-model Ψ is just the Heaviside function considered as a multivalued map, with jump at $s = 0$, i.e. we take the critical state to be equal to zero, which we may without loss of generality, by simply shifting the equation. In (1.1) we fix an open bounded domain $\mathscr{O} \subset \mathbb{R}^d$ and Δ denotes the Dirichlet Laplacian on $\mathscr{O}$, W_t is a cylindrical Wiener process on $L^2(\mathscr{O})$ with natural inner product $\langle \cdot, \cdot \rangle_2$, but the solution process X_t takes values in H defined to be the completion of $L^2(\mathscr{O})$ under the norm $\|\cdot\|_H$ corresponding to the inner product $\langle x, y\rangle_H := \langle x, (-\Delta)^{-1} y\rangle_2$, i.e. H is just the dual of the classical Sobolev space $H_0^{1,2}(\mathscr{O})$ (and is usually denoted by H^{-1}). To explain the type of noise in (1.1), let $\{e_k\}_{k\geq 1}$ be a normalized eigenbasis of $-\Delta$ in $L^2(\mathscr{O})$ with corresponding eigenvalues $\{\lambda_k\}_{k\geq 1}$ numbered in increasing orders with multiplicities. It is well-known that $\lambda_k = \mathrm{O}(k^{2/d})$ for large k. Then $B : H \to \mathscr{L}_2(L^2(\mathscr{O}), H)$ is defined by

$$B(x)h = \sum_{k=1}^{\infty} \mu_k \langle e_k, h\rangle_2 x e_k, \quad x \in H, h \in L^2(\mathscr{O}), \tag{1.3}$$

for $\{\mu_k\}_{k\geq 1} \subset \mathbb{R}$ chosen in such a way that $B(x) \in \mathscr{L}_2(L^2(\mathscr{O}), H), x \in H$, which is e.g. the case if there exists a constant $\varepsilon \in (0,1)$ such that (see (3.7) below) $\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{\frac{d}{2}\vee(1+\varepsilon)} < \infty$.

The fundamental question about (1.1) is now whether the system will go to the critical state (= 0 in our case) in finite time, i.e. letting

$$\tau_0 := \inf\{t \geq 0 : \ |X(t)|_H = 0\},$$

is $\tau_0 < \infty$ for any initial condition $X_0 = x \in H$, that is; do we have extinction in finite time?

Recently, it was proved in [5, 6] (where [6] is an improved version of [5]) that the answer is yes if $d = 1$, i.e. $\mathscr{O} \subset \mathbb{R}$, however, only with positive probability, that is, $\mathbb{P}(\tau_0 < \infty) > 0$, provided X_0 is not too far away from 0 (see [5, 6] for details). We mention here that in [5] the $\mu_k, k \geq 1$, introduced above, were assumed to be zero starting from $k \geq N+1$, an assumption that is dropped in [6]. Furthermore, in both [5, 6] for simplicity $d \leq 3$ was assumed. The question, however, whether

$$\mathbb{P}(\tau_0 < \infty) = 1 \tag{1.4}$$

was left open and it seemed to be out of reach of the methods in [5, 6].

The purpose of this paper is twofold. First, we develop a general technique to prove extinction for solutions of stochastic (ordinary and partial) differential equations, and second to apply the outcoming results to equations of type (1.1) (in fact for a whole class of $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$) to analyze the above question, more precisely the problem whether one of the following three properties hold:

(i) $\mathbb{P}(\tau_0 < \infty) > 0$ for small X_0,

(ii) $\mathbb{P}(\tau_0 < \infty) > 0$ for all X_0,

(iii) $\mathbb{P}(\tau_0 < \infty) = 1$ or, moreover, τ_0 has finite (exponential or polynomial) moments.

We already want to mention here that as a consequence we obtain that for all dimensions in the Zhang-model of SOC the extinction time τ_0 is even exponentially integrable, in particular we have extinction in finite time with probability one, provided the noise is small enough and $X_0 = x \in L^4(\mathscr{O}), x \geq 0$. This has been open even in the deterministic case. In order to also include stochastic fast diffusion equations and prove extinction for that case, strengthening and generalizing the results from [4] we study (1.1) for the following class of $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$: Ψ is a maximal monotone graph with $0 \in \Psi(0)$ such that for some constants $r \in [0, 1), C > 0, \theta_1 > 0, q \geq 1 + r$, and $\theta_2 \geq 0$

$$C(|s|^q + |s|) \geq sy \geq \theta_1 |s|^{1+r} + \theta_2 |s|^2, \quad s \in \mathbb{R}, y \in \Psi(s). \tag{1.5}$$

In this case we call (1.1) fast diffusion-type equation. A special case of this is the stochastic fast diffusion equation, i.e. equation (1.1) with

$$\Psi(s) = s^{r-1}|s|, \quad s \in \mathbb{R}, \tag{1.6}$$

where $r \in (0, 1)$. In this case we, however, avoid the assumption $d \leq 3$, made in [5] for simplicity, but prove extinction with positive probability in all dimensions with the usual dimension dependent restriction on r known from the deterministic case. As in [4, 5, 6] the latter is, of course, always included in our results choosing $\mu_k = 0$ for all $k \geq 1$ in (1.3).

Another new feature of this paper is that we prove our results for a whole class of general operators L on a measurable space replacing the Dirichlet Laplacian $(-\Delta)$ on $\mathscr{O} \subset \mathbb{R}^d$. This class, in particular, includes fractional Laplacians $L = (-\Delta)^\alpha, \alpha \in (0, 1]$, which have attracted more interest recently. Existence and uniqueness of solutions to (1.1), with $(-\Delta)$ replaced by a fractional Laplacian, have first been proved in [9] in the general, i.e. stochastic, hence including the deterministic ($B = 0$), case. In the deterministic case these results have been reproved in [12] under some restrictions on dimensions, however, by completely different methods.

The organization of this paper is as follows. In Section 2 we state and prove our two general results on extinction of solutions for stochastic equations. The first (see Theorem 2.1 below) gives quantitative conditions so that properties (i)-(iii) above hold respectively.

It also confirms the intuition that one gets stronger extinction results if one has more coercivity in the system (i.e. $\rho_2 > 0$ in condition $(H1)$ below or correspondingly $\theta_2 > 0$ in (1.5) above, as e.g. in the Zhang-model). The reason is that more coercivity means a stronger drift towards zero, so that the part in (1.5) with θ_1 in front becomes very big and pushes the process to zero. The second result (see Theorem 2.2 below) tells us that suitable large enough noise has the same effect. In Section 3 these results are applied to the fast diffusion-type equations above, but for the said general class of operators replacing $(-\Delta)$. Subsection 3.1 is devoted to the case, where $L = (-\Delta + V)^\alpha, \alpha \in (0, 1]$, and V a nonnegative measurable function. In Subsection 3.2 we prove exponential integrability of the extinction time τ_0 in the strongly dissipative case even for uncoloured noise (i.e. $\mu_k = 1$ for all $k \geq 1$ in (1.3)). In Subsection 3.3 we give examples of noises which lead to extinction with probability one.

The above described SOC case, i.e. Ψ is given by (1.2), and the fast diffusion case (1.6), both for the Laplacian and the fractional Laplacian, are considered as guiding examples, and are discussed in detail in Subsection 3.1.

2 A general result

Let $(E, \mathscr{E}, \mu)$ be a σ-finite measure space, and let $(L, \mathscr{D}(L))$ be a positive definite self-adjoint operator on $L^2(\mu)$ such that for its spectrum $\sigma(L)$ we have $\inf \sigma(L) > 0$. Let $P_t = \mathrm{e}^{-Lt}$. Let H be the completion of $L^2(\mu)$ w.r.t. the norm $\|\cdot\|_H$ corresponding to the inner product $\langle x, y\rangle_H := \langle x, L^{-1}y\rangle_2$ (note that since $\inf \sigma(L) > 0$, $L : \mathscr{D}(L) \to L^2(\mu)$ is bijective. So, this definition makes sense). For any $p, q \geq 1$, let $\|\cdot\|_p$ and $\|\cdot\|_{p\to q}$ the norm on $L^p(\mu)$ and the operator norm from $L^p(\mu)$ to $L^q(\mu)$ respectively.

Let $\{X_t\}_{t\geq 0}$ be an H-valued continuous adapted process on the complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Let

$$\tau_0 := \inf\{t \geq 0 : X_t = 0\}$$

be the extinction time of the process. To investigate the finiteness of τ_0, we introduce the following conditions:

$(H1)$ There exist $r \in [0, 1)$ and constants $\rho_1 > 0, \rho_2, \rho_3 \geq 0$ such that $\|X_t\|_H^2$ is a semi-martingale, $\|X_t\|_2^2$ and $\|X_t\|_{1+r}^{1+r}$ are locally integrable in t, and the Itô differential of $\|X_t\|_H^2$ satisfies

$$\mathrm{d}\|X_t\|_H^2 \leq 2\big\{\rho_3\|X_t\|_H^2 - \rho_1\|X_t\|_{1+r}^{1+r} - \rho_2\|X_t\|_2^2\big\}\mathrm{d}t + \mathrm{d}M_t$$

for some local martingale M_t.

$(H2)$ There exists $\theta \in (0, 1]$ such that

$$\gamma(\theta) := \int_0^\infty \mathrm{e}^{-\lambda_1(1-\theta)t}\|P_t\|_{1+r\to\frac{1+r}{r}}^\theta \mathrm{d}t < \infty,$$

where $\lambda_1 := \inf \sigma(L) > 0$ and $\sigma(L)$ is the spectrum of L.

Note that $(H1)$ implies that for any $X_0 \in L^2(\Omega \to H, \mathscr{F}_0, \mathbb{P})$ and $T > 0$, one has $(X_t)_{t\in[0,T]} \in L^{1+r}([0,T] \times \Omega \to L^{1+r}(\mu); \mathrm{d}t \times \mathbb{P})$ and furthermore, $(X_t)_{t\in[0,T]} \in L^2([0,T] \times \Omega \to L^2(\mu); \mathrm{d}t \times \mathbb{P})$ if $\rho_2 > 0$.

Remark 2.1. $(H1)$ implies that $\mathrm{e}^{-2\rho_3 t}\|X_t\|_H^2, t \geq 0$, is a non-negative local supermartingale. Hence it is equal to zero for all $t \geq \tau_0$ by a well-known result (see e.g. [8, Chap IV, Lemma 3.19]).

Remark 2.2. We would like to indicate that assumptions $(H1)$ and $(H2)$ are fulfilled for a large class of SPDEs. More precisely:

(1) Let e.g. $L = -\Delta + V$, where Δ is the Dirichlet Laplacian on an open domain $\mathscr{O} \subset \mathbb{R}^d$ and $V \geq 0$ is a continuous function on $\mathscr{O}$. If either $\mathscr{O}$ is bounded or $\liminf_{|x|\to\infty} V(x) > 0$, then $\lambda_1 > 0$ and

$$\|P_t\|_{1+r \to \frac{1+r}{r}} \leq c_1 \mathrm{e}^{-c_2 t} t^{-d(1-r)/[2(1+r)]}, \quad t > 0$$

holds for some constants $c_1, c_2 > 0$ (see Section 3 for details). Therefore, $(H2)$ holds for $\theta \in (0, \frac{2(1+r)}{d(1-r)}) \cap (0,1]$.

(2) Let X_t solve the following SPDE on H:

$$\mathrm{d}X_t + L\Psi(X_t) \ni B(X_t)\mathrm{d}W_t, \tag{2.1}$$

where W_t is a cylindrical Brownian motion on $L^2(\mu)$; $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph such that

$$C(|s|^{1+r} + |s|) \geq sy \geq \theta_1|s|^{1+r} + \theta_2|s|^2, \quad s \in \mathbb{R}, y \in \Psi(s)$$

for some constants $C > 0, \theta_1 > 0, \theta_2 \geq 0$; and $B : H \to \mathscr{L}_2(L^2(\mu); H)$ (the space of Hilbert-Schmidt linear operators from $L^2(\mu)$ to H) such that

$$\|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} \leq 2\rho_3\|x\|_H^2, \quad x \in H.$$

For what is meant by a solution to (2.1), we refer to Section 3 below. Then, by Itô's formula $(H1)$ holds (see Section 3 below).

Theorem 2.1. *Assume $(H1)$ and $(H2)$ and let $\mathbb{E}\|X_0\|_H^{\theta(1-r)} < \infty$.*

(1) *If either $\rho_2 > 0$ or $\rho_2 = 0$ but $\theta = 1$, then*

$$\mathbb{P}(\tau_0 = \infty) \leq 1 - \mathbb{E}\mathrm{e}^{-\theta(1-r)\rho_3\tau_0} \leq \frac{\rho_3\gamma(\theta)^{(1+r)/2}}{\rho_1^\theta \rho_2^{1-\theta} \lambda_1^{(1-r)(1-\theta)/2}} \mathbb{E}\|X_0\|_H^{\theta(1-r)},$$

where $\rho_2^{1-\theta} := 1$, if $\theta = 1$ and $\rho_2 = 0$. Consequently, $\mathbb{P}(\tau_0 < \infty) > 0$ holds for small $\mathbb{E}\|X_0\|_H^{\theta(1-r)}$.

(2) *If $\rho_3 < \lambda_1\rho_2$ then for any $\beta \in (0, \theta(1-r)(\rho_2\lambda_1 - \rho_3))$,*

$$\mathbb{E}e^{\beta\tau_0} \leq 1 + \frac{\beta\gamma(\theta)^{(1+r)/2}\mathbb{E}\|X_0\|_H^{\theta(1-r)}}{\theta(1-r)\rho_1^\theta\alpha_\beta^{1-\theta}\lambda_1^{(1-r)(1-\theta)/2}} < \infty$$

holds for $\alpha_\beta := \rho_2 - \frac{1}{\lambda_1}(\frac{\beta}{\theta(1-r)} + \rho_3) \in (0, \rho_2)$.

(3) *If $\rho_3 = \lambda_1\rho_2$ and $\theta = 1$, then*

$$\mathbb{E}\tau_0 \leq \frac{\gamma(1)^{(1+r)/2}}{(1-r)\rho_1}\mathbb{E}\|X_0\|_H^{1-r} < \infty.$$

Remark 2.3. If in $(H1)$ we have that $\rho_2 = 0$ and $M = 0$, which is e.g. the case for deterministic stochastic fast diffusion equations (see equation (3.4) with $B \equiv 0$ in Section 3 below), it follows by (1) (or (3)) in Theorem 2.1 that we have extinction in finite time. Thus we recover the well-known results from the deterministic case.

Obviously, in cases (2) and (3) of Theorem 2.1 one has $\mathbb{P}(\tau_0 < \infty) = 1$ for all deterministic initial data $X_0 = x \in H$, while in case (1) $\mathbb{P}(\tau_0 < \infty) > 0$ holds for small enough $X_0 = x \in H$. Our next result strengthens the assertion in (1): if the local martingale M_t is strong enough but not too large, then $\mathbb{P}(\tau_0 < \infty) > 0$ holds for all $X_0 = x \in H$.

Theorem 2.2. *Assume that $(H1)$ and $(H2)$ hold with either $\rho_2 > 0$, or $\rho_2 = 0$ but $\theta = 1$. If there exist two functions $g_2 \geq g_1 \in C([0, \infty))$ with $g_1(s) > 0$ for $s > 0$ such that*

$$g_2(\|X_t\|_H^2)\mathrm{d}t \geq \mathrm{d}\langle M\rangle_t \geq g_1(\|X_t\|_H^2)\mathrm{d}t. \tag{2.2}$$

define

$$\xi(s) := \frac{\rho_1^\theta\rho_2^{1-\theta}\lambda_1^{(1+r)(1-\theta)/2}}{\gamma(\theta)^{(1+r)/2}}s^{1-\theta(1-r)/2} - \rho_3 s, \quad s \geq 0,$$

where $\rho_2^{1-\theta} := 1$ if $\rho_2 = 0$ and $\theta = 1$; and set

$$g = g_2 1_{\{\xi \geq 0\}} + g_1 1_{\{\xi < 0\}}.$$

Then for any $X_0 = x \in H$,

$$\mathbb{P}(\tau_0 = \infty) \leq \frac{\int_0^{\|x\|_H^2} \exp\left[2\int_1^t \frac{\xi(u)}{g(u)}\mathrm{d}u\right]\mathrm{d}t}{\int_0^\infty \exp\left[2\int_1^t \frac{\xi(u)}{g(u)}\mathrm{d}u\right]\mathrm{d}t} < 1. \tag{2.3}$$

Consequently, if $g_1(s) \geq 2\rho_3 s^2$ holds for large s, then $\mathbb{P}(\tau_0 < \infty) = 1$.

Theorem 2.2 tells us that if the quadratic variation of M is big enough (but not too big, e.g. to avoid explosion in finite time), then we have extinction for all initial data in H. Intuitively, this means that in this case the process X will come close enough to zero with positive probability, so that by the strong Markov property Theorem 2.1 (1) applies.

To prove Theorems 2.1 and 2.2, we need the following two basic lemmas.

Lemma 2.3. $\|x\|_H \le \sqrt{\gamma(\theta)}\,\|x\|_2^{1-\theta}\|x\|_{1+r}^{\theta}, \quad x \in L^2(\mu) \cap L^{1+r}(\mu).$

Proof. Since by the symmetry of P_t and since $\inf \sigma(L) = \lambda_1$, we have

$$\mu(xP_t x) = \|P_{t/2}x\|_2^2 \le \mathrm{e}^{-\lambda_1 t}\|x\|_2^2,$$

and by the Hölder inequality

$$\mu(xP_t x) \le \|x\|_{1+r}\|P_t x\|_{(1+r)/r} \le \|x\|_{1+r}^2 \|P_t\|_{1+r \to \frac{1+r}{r}},$$

it follows that

$$\begin{aligned}\|x\|_H^2 &= \int_0^\infty \mu(xP_t x)\mathrm{d}t \\ &\le \|x\|_2^{2(1-\theta)}\|x\|_{1+r}^{2\theta}\int_0^\infty \mathrm{e}^{-\lambda_1(1-\theta)t}\|P_t\|_{1+r\to\frac{1+r}{r}}^{\theta}\mathrm{d}t = \gamma(\theta)\|x\|_2^{2(1-\theta)}\|x\|_{1+r}^{2\theta}.\end{aligned}$$

This completes the proof. □

Lemma 2.4. *For any $\theta \in (0,1]$,*

$$b^{(1-\theta)/\theta} + \frac{a}{b} \ge a^{1-\theta}, \quad b > 0, a \ge 0,$$

where $a^{1-\theta} := 1$ for $\theta = 1$.

Proof. If $b \ge a^\theta$ then

$$b^{(1-\theta)/\theta} + \frac{a}{b} \ge b^{(1-\theta)/\theta} \ge a^{1-\theta};$$

while if $b \le a^\theta$ then

$$b^{(1-\theta)/\theta} + \frac{a}{b} \ge \frac{a}{b} \ge a^{1-\theta}.$$

□

Proof of Theorem 2.1. By $(H1)$ and Itô's formula, we have

$$\begin{aligned}\mathrm{d}\|X_t\|_H^{\theta(1-r)} \le &-\theta(1-r)\|X_t\|_H^{\theta(1-r)-2}\big\{\rho_1\|X_t\|_{1+r}^{1+r} + \rho_2\|X_t\|_2^2 - \rho_3\|X_t\|_H^2\big\}\mathrm{d}t \\ &+ \mathrm{d}\tilde{M}_t, \quad t < \tau_0,\end{aligned} \tag{2.4}$$

where

$$\mathrm{d}\tilde{M}_t = \theta(1-r)\|X_t\|_H^{\theta(1-r)-2}\mathrm{d}M_t, \quad t < \tau_0.$$

Let $\alpha \in (0, \rho_2]$ and $\alpha = 0$ if $\rho_2 = 0, \theta = 1$. By Lemma 2.3 and Lemma 2.4 for

$$b := \frac{\|X_t\|_H^{1+r}}{\|X_t\|_2^{1+r}}, \quad a := \frac{\lambda_1^{(1-r)/2}\alpha\gamma(\theta)^{(1+r)/(2\theta)}\|X_t\|_H^{1-r}}{\rho_1},$$

and noting that $\|X_t\|_2^2 \geq \lambda_1 \|X_t\|_H^2$, we obtain for $t < \tau_0$

$$
\begin{aligned}
&\rho_1 \|X_t\|_{1+r}^{1+r} + \rho_2 \|X_t\|_2^2 - \rho_3 \|X_t\|_H^2 \\
&\geq \frac{\rho_1 \|X_t\|_H^{(1+r)/\theta}}{\gamma(\theta)^{(1+r)/(2\theta)} \|X_t\|_2^{(1-\theta)(1+r)/\theta}} + \alpha \|X_t\|_2^2 + \big\{(\rho_2 - \alpha)\lambda_1 - \rho_3\big\} \|X_t\|_H^2 \\
&\geq \frac{\rho_1 \|X_t\|_H^{1+r}}{\gamma(\theta)^{(1+r)/(2\theta)}} \left(b^{(1-\theta)/\theta} + \frac{a}{b} \right) + \big\{(\rho_2 - \alpha)\lambda_1 - \rho_3\big\} \|X_t\|_H^2 \qquad (2.5) \\
&\geq \frac{\rho_1 \|X_t\|_H^{1+r}}{\gamma(\theta)^{(1+r)/(2\theta)}} \left(\frac{\lambda_1^{(1-r)/2} \alpha \gamma(\theta)^{(1+r)/(2\theta)} \|X_t\|_H^{1-r}}{\rho_1} \right)^{1-\theta} + \big\{(\rho_2 - \alpha)\lambda_1 - \rho_3\big\} \|X_t\|_H^2 \\
&= \frac{\rho_1^\theta \alpha^{1-\theta} \lambda_1^{(1-r)(1-\theta)/2} \|X_t\|_H^{2-\theta(1-r)}}{\gamma(\theta)^{(1+r)/2}} + \big\{(\rho_2 - \alpha)\lambda_1 - \rho_3\big\} \|X_t\|_H^2.
\end{aligned}
$$

Let

$$
c_1 := \frac{\theta(1-r)\rho_1^\theta \alpha^{1-\theta} \lambda_1^{(1-r)(1-\theta)/2}}{\gamma(\theta)^{(1+r)/2}} > 0, \quad c_2 := \theta(1-r)\big\{(\rho_2 - \alpha)\lambda_1 - \rho_3\big\} \in \mathbb{R}.
$$

Combining (2.4) with (2.5) we obtain

$$
\mathrm{d}\|X_t\|_H^{\theta(1-r)} \leq -c_1 \mathrm{d}t - c_2 \|X_t\|_H^{\theta(1-r)} \mathrm{d}t + \mathrm{d}\tilde{M}_t, \quad t < \tau_0.
$$

Therefore,

$$
\mathrm{d}\{\|X_t\|_H^{\theta(1-r)} \mathrm{e}^{c_2 t}\} \leq -c_1 \mathrm{e}^{c_2 t} \mathrm{d}t + \mathrm{e}^{c_2 t} \mathrm{d}\tilde{M}_t, \quad t < \tau_0.
$$

This implies

$$
c_1 \mathbb{E} \int_0^{(\tau_0 \wedge t - \varepsilon)^+} \mathrm{e}^{c_2 s} \mathrm{d}s \leq \mathbb{E}\|X_0\|_H^{\theta(1-r)}, \quad t, \varepsilon > 0.
$$

Letting $\varepsilon \to 0$ and $t \to \infty$ we arrive at

$$
\frac{\mathbb{E}\mathrm{e}^{c_2 \tau_0} - 1}{c_2} = \mathbb{E} \int_0^{\tau_0} \mathrm{e}^{c_2 s} \mathrm{d}s \leq \frac{\mathbb{E}\|X_0\|_H^{\theta(1-r)}}{c_1}, \qquad (2.6)
$$

where $\frac{\mathbb{E}\mathrm{e}^{c_2 \tau_0} - 1}{c_2} := \mathbb{E}\tau_0$ if $c_2 = 0$. From this we are able to prove the desired assertions as follows.

(1) If either $\rho_2 > 0$ or $\rho_2 = 0$ but $\theta = 1$, we take $\alpha = \rho_2$. Then $c_1 > 0$ and $c_2 = -\theta(1-r)\rho_3 \leq 0$. By (2.6) we obtain

$$
\frac{\mathbb{E}\|X_0\|_H^{\theta(1-r)}}{c_1} \geq \frac{1 - \mathbb{E}\mathrm{e}^{-\theta(1-r)\rho_3 \tau_0}}{\theta(1-r)\rho_3},
$$

which implies the second inequality by the definition of c_1. The first inequality is trivial if $\rho_3 > 0$. When $\rho_3 = 0$ we have $c_2 = 0$, so that (2.6) implies that $\mathbb{E}\tau_0 < \infty$ and thus, the first inequality remains true.

(2) Let $\rho_3 < \lambda_1\rho_2$ and $\beta \in (0, \theta(1-r)(\lambda_1\rho_2 - \rho_3))$. Take $\alpha = \alpha_\beta$. We have $c_2 = \beta > 0$. So, (2.6) yields that

$$\mathbb{E}e^{\beta\tau_0} \le 1 + \frac{\beta\mathbb{E}\|X_0\|_H^{\theta(1-r)}}{c_1} = 1 + \frac{\beta\gamma(\theta)^{(1+r)/2}\mathbb{E}\|X_0\|_H^{\theta(1-r)}}{\theta(1-r)\rho_1^\theta\alpha_\beta^{1-\theta}\lambda_1^{(1-r)(1-\theta)/2}}.$$

(3) If $\rho_3 = \lambda_1\rho_2$ and $\theta = 1$, we take $\alpha = 0$ so that $c_1 > 0$ and $c_2 = 0$. Therefore, (2.6) implies that

$$\mathbb{E}\tau_0 \le \frac{\mathbb{E}\|X_0\|_H^{1-r}}{c_1} = \frac{\gamma(1)^{(1+r)/2}}{(1-r)\rho_1}\mathbb{E}\|X_0\|_H^{1-r}.$$

$\square$

Proof of Theorem 2.2. By (2.5) with $\alpha = \rho_2$ and $(H1)$ we have for $t < \tau_0$

$$\mathrm{d}\|X_t\|_H^2 \le -\xi(\|X_t\|_H^2)\mathrm{d}t + \mathrm{d}M_t. \tag{2.7}$$

By the definition of ξ we see that there exists a constant $r_0 > 0$ such that ξ is strictly positive and increasing on $(0, r_0]$, and

$$\int_0^{r_0} \frac{1}{\xi(t)}\mathrm{d}t < \infty. \tag{2.8}$$

For any constant $N > r_0 \vee \|x\|_H^2$, let

$$\tau_N := \inf\{t \ge 0 : \|X_t\|_H^2 > N\}$$

and

$$f_N(s) := \int_0^s \mathrm{d}t \int_t^N \frac{2}{g_1(u)} \exp\left[-2\int_t^u \frac{\xi(v)}{g_1(v)}\,\mathrm{d}v\right]\mathrm{d}u, \quad s \in (0, N].$$

Since ξ is strictly positive and increasing on $(0, r_0]$, for $t \in (0, r_0]$ one has

$$\begin{aligned}&\int_t^{r_0} \frac{2}{g_1(u)} \exp\left[-2\int_t^u \frac{\xi(v)}{g_1(v)}\,\mathrm{d}v\right]\mathrm{d}u\\ &\le \frac{1}{\xi(t)}\int_t^{r_0} \frac{2\xi(u)}{g_1(u)} \exp\left[-2\int_t^u \frac{\xi(v)}{g_1(v)}\,\mathrm{d}v\right]\mathrm{d}u \le \frac{1}{\xi(t)}.\end{aligned}$$

Combining this with (2.8) we conclude that $f_N \in C^2((0, N])$. Moreover, it is easy to check that $f_N'' \le 0$ and

$$-\xi(s)f_N'(s) + \frac{g_1(s)}{2}f_N''(s) = -1, \quad s \in (0, N].$$

Thus, by (2.7), $\mathrm{d}\langle M\rangle_t \ge g_1(\|X_t\|_H^2)\mathrm{d}t$ and Itô's formula,

$$\mathrm{d}f_N(\|X_t\|_H^2) \le -\mathrm{d}t + f_N'(\|X_t\|_H^2)\mathrm{d}M_t, \quad t < \tau_0 \wedge \tau_N.$$

Therefore,

$$\mathbb{E}(\tau_0 \wedge \tau_N) \le f_N(\|x\|_H^2) < \infty. \tag{2.9}$$

On the other hand, taking

$$f(s) := \int_0^s \exp\left[2\int_1^t \frac{\xi(u)}{g(u)}\mathrm{d}u\right]\mathrm{d}t, \quad s \ge 0,$$

we have

$$-\xi(s)f'(s) + \frac{g(s)}{2}f''(s) = 0, \quad s > 0.$$

Moreover, by assumption and the definition of g

$$f''(\|X_t\|_H^2)\mathrm{d}\langle M\rangle_t \le f''(\|X_t\|_H^2)g(\|X_t\|_H^2)\mathrm{d}t.$$

Therefore, by (2.7), $f(\|X_t\|_H^2)$ is a super-martingale up to time $\tau_0 \wedge \tau_N$. So,

$$f(\|x\|_H^2) \ge \mathbb{E}f(\|X_{t\wedge\tau_0\wedge\tau_N}\|_H^2) \ge \mathbb{P}(\tau_N \le t \wedge \tau_0)f(N), \quad t > 0.$$

Combining this with (2.9) we obtain

$$\mathbb{P}(\tau_0 = \infty) = \mathbb{P}(\tau_0 = \infty, \tau_N \wedge \tau_0 < \infty) \le \lim_{t\to\infty}\mathbb{P}(\tau_N \le \tau_0 \wedge t) \le \frac{f(\|x\|_H^2)}{f(N)}.$$

Then the first part of the assertion follows by letting $N \to \infty$. The second follows by realizing that the denominator of the right-hand side of (2.3) is equal to infinity if $g(s) \ge \rho_3 s^2$ for large s, and that for large s one has $\xi(s) < 0$ and hence $g(s) = g_1(s)$. □

3 Applications to stochastic fast-diffusion equations

The aim of this section is to apply Theorems 2.1 and 2.2 to a class of SPDEs as mentioned in Remark 2.2 and in the Introduction.

Let $(E, \mathscr{E}, \mu)$, L and P_t be as in Section 2 such that the spectrum of L is discrete with strictly positive eigenvalues $\{\lambda_k\}_{k\ge1}$ counting multiplicities with increasing order, and let $\{e_k\}_{k\ge1}$ be the corresponding unit eigenfunctions forming an ONB of $L^2(\mu)$. Then H is the completion of $L^2(\mu)$ w.r.t. the inner product

$$\langle x, y\rangle_H := \langle x, L^{-1}y\rangle_2 = \sum_{k=1}^\infty \frac{\langle x, e_k\rangle_2\langle y, e_k\rangle_2}{\lambda_k},$$

where $\langle\cdot,\cdot\rangle_2$ also denotes both the inner product in $L^2(\mu)$ and its extension to $H \times H^*$. Next, let

$$\|P_t\|_{1\to\infty} \le (c_\infty t)^{-d/2}, \quad t > 0 \tag{3.1}$$

hold for some constants $d, c_\infty \in (0,\infty)$. By (3.1), $\|P_t\|_{2\to 2} \le \mathrm{e}^{-\lambda_1 t}$, and the Riesz-Thorin interpolation theorem, we have

$$\|P_t\|_{1+r\to\frac{1+r}{r}} \le (c_\infty t)^{-\frac{d(1-r)}{2(1+r)}} \mathrm{e}^{-\lambda_1(1-r)t/(1+r)}, \quad t>0.$$

Therefore,

$$\gamma(\theta)<\infty, \ \text{ if } \ \theta \in \Big(0, \frac{2(1+r)}{d(1-r)}\Big) \cap (0,1]. \tag{3.2}$$

A standard example for the framework is that $L=-\Delta+V$ for the Dirichlet Laplacian Δ on a domain $\mathscr{O}\subset\mathbb{R}^d$ having finite volume, and for a nonnegative locally bounded measurable function V on $\mathscr{O}$. It is also the case if $\mathscr{O}$ has infinite volume but

$$\mu(V\le r):=\mu(\{x\in\mathscr{O}: V(x)\le r\})<\infty, \ \ r\ge 0,$$

where μ stands for the Lebesgue measure on $\mathscr{O}$. In this case, according to [11], L has discrete spectrum as well.

Moreover, let $\Psi:\mathbb{R}\to 2^{\mathbb{R}}$ be a maximal monotone graph such that $0\in\Psi(0)$ and

$$C(|s|^q+|s|)\ge sy\ge \theta_1|s|^{1+r}+\theta_2 s^2, \ \ s\in\mathbb{R}, y\in\Psi(s) \tag{3.3}$$

holds for some constants $r\in[0,1), C>0, \theta_1>0, \theta_2\ge 0, q\ge 1+r$.

Now, let W_t be a cylindrical Brownian motion on $L^2(\mu)$. Extending the framework investigated in [4, 5, 6], where $L=-\Delta$ on a bounded domain in $\mathbb{R}^d$ (for $d\le 3$), we consider the following SPDE on H:

$$\mathrm{d}X_t + L\Psi(X_t)\mathrm{d}t \ni B(X_t)\mathrm{d}W_t, \tag{3.4}$$

where $B: H\to \mathscr{L}_2(L^2(\mu);H)$ is measurable, subject to conditions to be specified in the following Subsections 3.1-3.3.

Following Definition 2.1 in [5] we call a continuous adapted process $X:=(X_t)_{t\ge 0}$ on H a solution to (3.4) if:

(a) $(X_t)_{t\in[0,T]}\in L^2([0,T]\times\Omega\to L^2(\mu);\mathrm{d}t\times\mathbb{P})$ for any $T>0$.

(b) There exists a progressively measurable process $\eta:=(\eta_t)_{t\ge 0}$ such that $(\eta_t)_{t\in[0,T]}\in L^2([0,T]\times\Omega\to L^2(\mu);\mathrm{d}t\times\mathbb{P})$ for any $T>0$, $\eta\in\Psi(X)$ $\mathrm{d}t\times\mathbb{P}\times\mu$-a.e., and $\mathbb{P}$-a.s.

$$\langle X_t, e_k\rangle_H = \langle X_0, e_k\rangle_H - \int_0^t \langle \eta_s, e_k\rangle_2 \mathrm{d}s + \int_0^t \langle B(X_s)\mathrm{d}W_s, e_k\rangle_H, \ \ t\ge 0, k\ge 1.$$

Applying Itô's formula to $\langle X_t, e_k\rangle_H^2$, multiplying by λ_k and summing over $k\in\mathbb{N}$, it follows that

$$\mathrm{d}\|X_t\|_H^2 = -2\langle X_t,\eta_t\rangle_2\mathrm{d}t + \|B(X_t)\|^2_{\mathscr{L}_2(L^2(\mu);H)}\mathrm{d}t + 2\langle B(X_t)\mathrm{d}W_t, X_t\rangle_H. \tag{3.5}$$

3.1 Extinction for stochastic fast-diffusion type equations with linear multiplicative noise

In this subsection we consider $L = (-\Delta + V)^\alpha$ for Δ the Dirichlet Laplacian on a domain $\mathscr{O} \subset \mathbb{R}^n$ and $V \geq 0$ being a measurable function on $\mathscr{O}$ such that the spectrum of L is discrete, where $n \in \mathbb{N}$ and $\alpha \in (0, 1]$ are fixed constants. Let μ be the Lebesgue measure on $\mathscr{O}$. It is well-known that the semigroup $P_t^{(0)} := \mathrm{e}^{t(\Delta - V)}$ satisfies

$$\|P_t^{(0)}\|_{1\to\infty} \leq c_0 t^{-n/2}, \quad t > 0$$

for some constant $c_0 \in (0, \infty)$. Then (3.1) holds for $P_t := \mathrm{e}^{-tL}$ with $d = \frac{n}{\alpha}$ and some constant $c_\infty \in (0, \infty)$. Consider (3.4) for

$$B(x)h = \sum_{k=1}^{\infty} \mu_k \langle h, e_k \rangle_2 x e_k, \quad x \in L^2(\mu), h \in L^2(\mu), \tag{3.6}$$

where $\{\mu_k\}_{k\geq 1}$ is a sequence of constants such that

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{\frac{d}{2} \vee (1+\varepsilon)} < \infty \tag{3.7}$$

holds for some constant $\varepsilon > 0$. When $L = -\Delta$ for Δ the Dirichlet Laplacian on a bounded domain in $\mathbb{R}^d$,

$$\lambda_k = \mathrm{O}(k^{2/d})$$

holds for large k, so that (3.7) follows from

$$\sum_{k=1}^{\infty} \mu_k^2 k^{1 \vee (\frac{2}{d} + \varepsilon)} < \infty$$

for some $\varepsilon > 0$. We note that when $d \neq 2$ (3.7) is weaker than the corresponding condition used in [6] (in [4, 5] the condition is even stronger as only finite modes of the noise are allowed).

To ensure that $(H1)$ holds for solutions to (3.4), we first observe that (3.7) implies $\|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} \leq \rho_3 \|x\|_H^2$ for some constant $\rho_3 > 0$.

Proposition 3.1. *Let B be as in (3.6). If (3.7) holds, then the linear map $L^2(\mu) \ni x \mapsto B(x) \in \mathscr{L}_2(L^2(\mu), H)$ extends by continuity to all of H and*

$$\rho_3 := \frac{1}{2} \sup_{\|x\|_H^2 = 1} \|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} < \infty, \tag{3.8}$$

so that $\|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} \leq 2\rho_3 \|x\|_H^2$ holds for all $x \in H$.

To prove this result, we need the following lemma.

Lemma 3.2. *Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a symmetric Dirichelt form on $L^2(\mu)$ over a σ-finite measure space $(E, \mathscr{F}, \mu)$, and let $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$ be the associated Dirichlet operator, i.e. the negative definite self-adjoint operator on $L^2(\mu)$ associated to the symmetric form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. Then for any $g \in L^\infty(\mu) \cap \mathscr{D}(\mathscr{L})$ such that $Lg \in L^\infty(\mu)$,*

$$\mathscr{E}(fg, fg) \le \|g\|_\infty^2 \mathscr{E}(f, f) - \mu(f^2 g \mathscr{L} g), \quad f \in \mathscr{D}(\mathscr{E}).$$

Proof. For $t > 0$, let J_t be the symmetric measure on $E \times E$ such that

$$J_t(A \times B) = \mu(1_A P_t 1_B), \quad A, B \in \mathscr{F},$$

where P_t is the associated Markov semigroup. Then, by the symmetry of J_t,

$$\begin{aligned}
\mathscr{E}(h,h) &= \lim_{t\downarrow 0} \frac{1}{t} \int_E h(h - P_t h) \mathrm{d}\mu = \lim_{t\downarrow 0} \frac{1}{t} \int_{E\times E} h(x) \Big(\frac{h}{P_t 1}(x) - h(y) \Big) J_t(\mathrm{d}x, \mathrm{d}y) \\
&= \lim_{t\downarrow 0} \frac{1}{t} \bigg(\int_{E\times E} h(x)\big(h(x) - h(y)\big) J_t(\mathrm{d}x, \mathrm{d}y) + \int_T h^2(1 - P_t 1) \mathrm{d}\mu \bigg) \\
&= \lim_{t\downarrow 0} \frac{1}{t} \bigg(\frac{1}{2} \int_{E\times E} \Big\{ h(x) - h(y) \Big\}^2 J_t(\mathrm{d}x, \mathrm{d}y) + \int_T h^2(1 - P_t 1) \mathrm{d}\mu \bigg), \quad h \in \mathscr{D}(\mathscr{E}).
\end{aligned}$$

Combining this with

$$\Big\{ (fg)(x) - (fg)(y) \Big\}^2 = (f(x) - f(y))^2 g(x) g(y) + \big(f(y)^2 g(y) - f(x)^2 g(x) \big) \big(g(y) - g(x) \big),$$

and using the symmetry of J_t, we arrive at

$$\begin{aligned}
&\mathscr{E}(fg, fg) = \lim_{t\downarrow 0} \frac{1}{t} \bigg(\frac{1}{2} \int_{E\times E} \Big\{ (fg)(x) - (fg)(y) \Big\}^2 J_t(\mathrm{d}x, \mathrm{d}y) + \int_T f^2 g^2 (1 - P_t 1) \mathrm{d}\mu \bigg) \\
&\le \lim_{t\downarrow 0} \frac{1}{t} \int_{E\times E} \Big(\frac{\|g\|_\infty^2}{2} \big\{ f(x) - f(y) \big\}^2 - f(x)^2 g(x) \big(g(y) - g(x) \big) \Big) J_t(\mathrm{d}x, \mathrm{d}y) \\
&= \lim_{t\downarrow 0} \frac{1}{t} \bigg\{ \frac{\|g\|_\infty^2}{2} \int_{E\times E} \Big\{ f(x) - f(y) \Big\}^2 J_t(\mathrm{d}x, \mathrm{d}y) - \int_E \Big\{ f^2 g (P_t g - g) + f^2 g^2 (1 - P_t 1) \Big\} \mathrm{d}\mu \bigg\} \\
&\le \|g\|_\infty^2 \mathscr{E}(f, f) - \mu(f^2 g \mathscr{L} g).
\end{aligned}$$

□

Proof of Proposition 3.1. First we note that by (3.1),

$$\|e_k\|_\infty = \mathrm{e} \|P_{1/\lambda_k} e_k\|_\infty \le \mathrm{e} \|P_{1/\lambda_k}\|_{2\to\infty} \le c' \lambda_k^{d/4} \tag{3.9}$$

holds for some constant $c' > 0$. Since $B(x)$ is linear in x, it suffices to prove (3.8). By the definition of $B(x)$, we have

$$\|B(x)\|_{\mathscr{L}_2(L^2(\mu);H)}^2 = \sum_{k=1}^\infty \|B(x) e_k\|_H^2 = \sum_{k=1}^\infty \mu_k^2 \|x e_k\|_H^2. \tag{3.10}$$

So, it suffices to show that there exists a constant $c > 0$ such that

$$\|xe_k\|_H^2 \le c\|x\|_H^2 \lambda_k^{\frac{d}{2}\vee(1+\varepsilon)}, \quad x \in H, k \ge 1. \tag{3.11}$$

By (3.9) and an approximation argument, we only need to prove this inequality for $x \in L^2(\mu)$. In this case

$$\|xe_k\|_H = \sup_{\mathscr{E}(f,f)\le 1} \mu(xe_k f) \le \|x\|_H \sup_{\mathscr{E}(f,f)\le 1} \sqrt{\mathscr{E}(e_k f, e_k f)}, \tag{3.12}$$

where $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is the associated Dirichlet form associated to $\mathscr{L} := -L$. Since $\langle f^2 e_k, Le_k\rangle_2 = \lambda_k \mu(f^2 e_k^2)$, by Lemma 3.2 with $\mathscr{L} = -L$, we obtain

$$\mathscr{E}(e_k f, e_k f) \le \|e_k\|_\infty^2 + \lambda_k \mu(e_k^2 f^2). \tag{3.13}$$

For $p := d \vee (2 + \frac{\varepsilon}{d})$, it follows from (3.1) and $\|P_t\|_2 \le \mathrm{e}^{-\lambda_1 t}$ that

$$\|P_t\|_{1\to\infty} \le ct^{-p/2}, \quad t > 0$$

holds for some constant $c > 0$. This implies the Sobolev inequality (cf. [7])

$$\|f\|_{\frac{2p}{p-2}}^2 \le C\mathscr{E}(f,f), \quad f \in \mathscr{D}(\mathscr{E})$$

for some constant $C > 0$. Combining this with (3.13) and noting that $\|e_k\|_2 = 1$, we obtain

$$\mathscr{E}(e_k f, e_k f) \le \|e_k\|_\infty^2 + C\lambda_k \|e_k\|_p^2 \le \|e_k\|_\infty^2 + C\lambda_k \|e_k\|_\infty^{\frac{2(p-2)}{p}}. \tag{3.14}$$

Combining (3.9) with (3.12) and (3.14) we prove (3.11). □

Now exactly the same arguments as in the proof of [5, Theorem 2.2] imply that for any $x \in L^{4\vee(2q)}(\mu)$, there exists a unique solution to (3.4) with $X_0 = x$, which is non-negative if $x \ge 0$. Hence by Proposition 3.1 and (3.5), we have the following consequence of Theorem 2.1:

Corollary 3.3. *Assume $X_0 = x \in L^{4\vee 2q}(\mathscr{O})$ and that (3.3) and (3.7) hold. Let $\rho_1 = \theta_1, \rho_2 = \theta_2$, and let ρ_3 be as in Proposition 3.1. Then for any $\theta \in \big(0, \frac{2(1+r)}{d(1-r)}\big) \cap (0,1], d := \frac{n}{\alpha}$, all assertions in Theorem 2.1 hold for solutions to (3.4) with B given in (3.12).*

Example 3.1. (i) (SOC-case). Let Ψ be as in (1.2). In this case (3.3) holds with $r = 0$, some $\theta_1 > 0, C > 0$ and $q = 1, \theta_2 = 0$ in the BTW-model and $q = 2, \theta_2 > 0$ in the Zhang-model respectively for $s \ge 0$. But if $X_0 = x \in L^4(\mathscr{O}), x \ge 0$, as mentioned above, we have $X_t \ge 0$ for all $t \ge 0$. Therefore, to consider $s \ge 0$ is sufficient, since we may change Ψ on $(-\infty, 0)$ to become an odd function without changing anything. Hence we can apply Proposition 3.1 with B as in (3.6) satisfying (3.7). In the BTW-model $\theta_2 = 0$, so we need $\frac{2\alpha}{n} > 1$, i.e. $n = 1$ and $\alpha > \frac{1}{2}$, and we have only extinction in finite time with

positive probability for small enough noise or initial conditions $X_0 = x \in L^4(\mathscr{O}), x \geq 0$, with small enough H-norm. So, we recover the corresponding result from [5, 6] in the special case $\alpha = 1$. Furthermore, if $B = 0$, hence $\rho_3 = 0$ in Theorem 2.1(3), which thus applies for $n = 1, \alpha > \frac{1}{2}$, to give extinction, recovering the deterministic case from [5, 6]. In the Zhang-model, however, we have $\rho_2 = \theta_2 > 0$. Hence for all $\alpha \in (0, 1]$ and all dimensions n, if we choose $\theta \in (0, \frac{2\alpha}{n})$, and for small enough noise we can apply Theorem 2.1(2) to get exponential integrability of the extinction time, hence in particular extinction in finite time with probability one, provided $X_0 = x \in L^4(\mathscr{O}), x \geq 0$. In particular, for the deterministic Zhang-model we have $r = \rho_3 = 0$ and $\rho_1 = \theta_1, \rho_2 = \theta_2 > 0$, so that Theorem 2.1(2) implies that

$$\tau_0 = \lim_{\beta \to 0} \frac{1}{\beta}(\mathrm{e}^{\beta \tau_0} - 1) \leq \inf_{\theta \in (0,1]} \frac{\gamma(\theta)^{\frac{1}{2}} \|X_0\|_H^\theta}{\theta \theta_1^\theta \theta_2^{1-\theta} \lambda_1^{\frac{1-\theta}{2}}} < \infty.$$

(ii) (Fast diffusion-case). Let Ψ be as in (1.6). Then (3.3) holds with $\theta_1 = 1, C = 1, q = 1 + r, \theta_2 = 0$. So, if B is not identically equal to zero, only Theorem 2.1(1) applies with $\rho_2 = \theta_2 = 0$ and $\theta = 1$. So, we need $\frac{2\alpha(1+r)}{n(1-r)} > 1$, i.e. $r > \frac{n-2\alpha}{n+2\alpha}$, and we only have extinction in finite time with positive probability for small enough noise or initial conditions with small enough H-norm. So, again we recover the corresponding results from [4, 6] in the special case $\alpha = 1, n \leq 3$ (though for $n = 3, \alpha = 1$, also the case $r = \frac{n-2\alpha}{n+2\alpha} + \frac{1}{5}$ is covered in [4, 6]). If $B = 0$, Theorem 2.1(3) applies with $\theta = 1$, leading to the same restriction $r > \frac{n-2\alpha}{n+2\alpha}$. Hence we get extinction in finite time for the deterministic case, which appears, however, to be a new result if $\alpha < 1$. For $\alpha = 1$ we recover the well-known results for the deterministic fast diffusion equation. Finally, we point out, that, adding a linear term to Ψ we again get extinction in finite time with probability one in the same way as in the Zhang-model above.

In the next subsection, we consider much stronger noises such that $\mu_k = 1$ in (3.6) is allowed.

3.2 Exponential integrability of τ_0 for strongly dissipative equations with uncoloured linear multiplicative noise

In addition to (3.3), we assume that

$$\Psi \in C(\mathbb{R}), \quad (s-t)(\Psi(s) - \Psi(t)) \geq \kappa |s-t|^2, \qquad s, t \in \mathbb{R} \tag{3.15}$$

holds for some constants $\kappa > 0$. Because $\Psi(0) = 0$, (3.15) implies that (3.3) holds with $\theta_2 > 0$. A simple example of Ψ satisfying (3.3) and (3.15) is

$$\Psi(s) = \theta_1 |s|^r \mathrm{sgn}(s) + \theta_2 s$$

for some constants $r \in (0, 1), \theta_1, \theta_2 > 0$. Let B_0 be a bounded linear operator on $L^2(\mu)$. Let

$$\rho_0 := \frac{1}{2} \|B_0\|_{2\to2}^2 \sum_{k=1}^\infty \frac{\|e_k\|_\infty^2}{\lambda_k} \in [0, \infty],$$

where $\|B_0\|_{2\to 2}$ is the operator norm of B_0 in $L^2(\mu)$. We consider the following stochastic differential equation on H:

$$\mathrm{d}X_t = -L\Psi(X_t)\mathrm{d}t + X_t B_0 \mathrm{d}W_t. \tag{3.16}$$

It will turn out that if $\rho_0 < \infty$, then this equation is a special case of (3.4), since in this case the operator-valued function B defined by

$$B(x)h = xB_0h, \quad h \in L^2(\mu),\ x \in L^\infty(\mu)$$

extends by continuity to all $x \in L^2(\mu)$ (see (3.17) below) which is sufficient under condition (3.15). We emphasize here that $\rho_0 < \infty$, e.g. if L is the Dirichlet Laplacian on $(0,1)$ (cf. Remark 3.1 below). Since Ψ is single-valued we can use the result on uniqueness and existence of solutions from [9], which holds even for random initial conditions, as then does the following theorem.

Theorem 3.4. *Assume that (3.1), (3.3), (3.15) hold. If $\rho_0 \in (0,\kappa] \cap (0,\theta_2)$, then for any $X_0 \in L^2(\Omega \to H, \mathscr{F}_0; \mathbb{P})$, the equation (3.16) has a unique solution in the sense of [9]. If moreover (3.1) holds, then for any $\theta \in \big(0, \frac{2(1+r)}{d(1-r)}\big) \cap (0,1]$ and any $\beta \in \big(0, \theta(1-r)(\theta_2 - \rho_0)\lambda_1\big)$,*

$$\mathbb{E}\mathrm{e}^{\beta\tau_0} \le 1 + \frac{\beta\gamma(\theta)^{(1+r)/2}\mathbb{E}\|X_0\|_H^{\theta(1-r)}}{\theta(1-r)\theta_1^\theta\{\theta_2 - \beta/(\lambda_1\theta(1-r))\}^{1-\theta}\lambda_1^{(1-r)(1-\theta)/2}} < \infty$$

provided $\mathbb{E}\|X_0\|_H^{\theta(1-r)} < \infty$.

Proof. For the existence and uniqueness of solutions, we need only to verify the assumptions of [9, Theorem 2.1]. To this end, we take $\mathbf{V} = L^2(\mu)$, $A(x) = -L\Psi(x)$, $R(x) = \mu(x^2)$ and for fixed $T > 0$, $K = L^2([0,T]\times\Omega\times E; \mathrm{d}t \times \mathbb{P} \times \mu)$. Then assumptions **(K)**, **(H1)** and **(H4)** in [9] follow immediately from (3.3) and the continuity of Ψ. It remains to verify that for some constants $c_1, c_2, c_4 \in \mathbb{R}$ and $c_3 > 0$

(H2) $-2\langle\Psi(x) - \Psi(y), x-y\rangle_2 + \|B(x) - B(y)\|^2_{\mathscr{L}_2(L^2(\mu);H)} \le c_1\|x-y\|_H^2, \quad x,y \in L^2(\mu);$

(H3) $-2\langle\Psi(x), x\rangle_2 + \|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} \le c_2\|x\|_H^2 - c_3\|x\|_2^2 + c_4, \quad x \in L^2(\mu).$

Observe that for $x \in L^\infty(\mu)$

$$\begin{aligned}\|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} &= \sum_{k=1}^\infty \|xB_0e_k\|_H^2 = \sum_{k,j=1}^\infty \frac{\mu(\{B_0e_k\}e_jx)^2}{\lambda_j} = \sum_{k,j=1}^\infty \frac{\mu(e_kB_0^*(e_jx))^2}{\lambda_j}\\ &= \sum_{j=1}^\infty \frac{\|B_0^*(e_jx)\|_2^2}{\lambda_j} \le \|B_0\|^2_{2\to 2}\|x\|_2^2\sum_{j=1}^\infty\frac{\|e_j\|_\infty^2}{\lambda_j} = 2\rho_0\|x\|_2^2.\end{aligned} \tag{3.17}$$

Since $B(x) - B(y) = B(x-y)$, combining (3.17) with (3.15) and noting that $\kappa \ge \rho_0$, we obtain **(H2)** for $c_1 = 0$. Moreover, (3.3) and (3.17) imply that **(H3)** holds for $c_2 = c_4 = 0$ and $c_3 = \theta_2 - \rho_0 > 0$. Next, by (3.5), we have

$$\mathrm{d}\|X_t\|_H^2 \le -2\big\{\theta_1\|X_t\|_{1+r}^{1+r} + \theta_2\|X_t\|_2^2 - \rho_0\|X_t\|_2^2\big\}\mathrm{d}t + \mathrm{d}M_t,$$

where

$$\mathrm{d}M_t = 2\langle X_t B_0 \mathrm{d}W_t, X_t\rangle_H.$$

This implies $(H1)$ for $\rho_1 = \theta_1, \rho_2 = \theta_2 - \rho_0 > 0$ and $\rho_3 = 0$. Then the desired result on τ_0 follows from Theorem 2.1(2). $\square$

Remark 3.1. Let e.g. $L = -\Delta$ on $(0,1)$. Then we have

$$\lambda_k = \pi^2 k^2, \quad e_k(s) = \sqrt{2}\sin(\pi k s), \quad k \geq 1, s \in (0,1),$$

so that $\rho_0 < \infty$. Therefore, $\rho_0 < \infty$ and Theorem 3.4 applies, for small enough $\|B_0\|_{2\to 2}$.

In the next subsection, we consider the case with a noise having a component in the direction X_t, so that Theorem 2.2 applies.

3.3 Extinction with probability 1 for special noise

Now let us consider again the situation of Subsection 3.1. Let e be an unit element in $L^2(\mu)$ and let

$$B(x)h = cx\langle h, e\rangle_2, \quad x \in H, h \in L^2(\mu),$$

where $c \neq 0$ is a constant. Taking $B_t := \langle W_t, e\rangle_2$ (a one-dimensional Brownian motion) (3.4) reduces to

$$\mathrm{d}X_t + L\Psi(X_t) \ni cX_t \mathrm{d}B_t. \tag{3.18}$$

By (3.3) and (3.5), $(H1)$ holds for $\rho_1 = \theta_1, \rho_2 = \theta_2$ and

$$\mathrm{d}M_t = 2c\langle X_t, X_t\rangle_H \mathrm{d}t.$$

We have

$$\mathrm{d}\langle M\rangle_t = 4c^2\|X_t\|_H^4 \mathrm{d}t = g(\|X_t\|_H^2)\mathrm{d}t$$

for $g(s) = 4c^2 s^2$.

Corollary 3.5. *Assume that (3.1) and (3.3) hold, and let $\rho_3 = \frac{c^2}{2}$. If either $\theta_2 > 0$ or $\theta_2 = 0$ but $\frac{2(1+r)}{d(1-r)} > 1$, then $\mathbb{P}(\tau_0 < \infty) = 1$ holds for solutions to (3.19) for any $X_0 = x \in H$.*

Proof. Obviously, $g(s) \geq 2\rho_3 s^2$ and $(H1)$ hold. By Theorem 2.2 for $g_1 = g_2 = g$, it suffices to note that due to (3.2) one has $\gamma(\theta) < \infty$ for all $\theta \in \big(0, \frac{2(1+r)}{d(1-r)}\big) \cap (0,1]$, and thus $\gamma(1) < \infty$ if $\frac{2(1+r)}{d(1-r)} > 1$. $\square$

Remark 3.2. We note that to make stochastic perturbations in directions other than X_t, we may consider

$$\mathrm{d}X_t + L\Psi(X_t) \ni cX_t\mathrm{d}B_t + \tilde{B}(X_t)\mathrm{d}\tilde{W}_t, \tag{3.19}$$

where $\tilde{W}_t$ be a cylindrical Brownian motion on $L^2(\mu)$ which is independent of W_t, and $\tilde{B} : H \to \mathscr{L}_2(L^2(\mu); H)$ is such that $\|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} \le \tilde{c}^2\|x\|^2_H$. Then Theorem 2.1 applies to $\rho_3 = \frac{1}{2}(c^2 + \tilde{c}^2)$, while the assertion in Theorem 2.2 holds for this ρ_3 and $g_1(s) = 4c^2s^2, g_2(s) = 4(c^2 + \tilde{c}^2)s^2$.

Finally, we consider one more case which in fact generalizes the one considered above (take $N = 0$ below), but for, which the noise exists not only in one, but in finitely many directions, and both Theorem 2.1 and Theorem 2.2 apply: Let $N \in \mathbb{N}$ and consider (3.4) for

$$B(x)h = \sum_{k=1}^{N} \mu_k\langle x, e_k\rangle_2\langle h, e_k\rangle_2 e_k + \mu_{N+1}\langle h, e_{N+1}\rangle_2\pi_N^{\perp}(x), \quad x, h \in H, \tag{3.20}$$

where $\{\mu_k\}_{k=1}^{N+1} \subset \mathbb{R}$ and

$$\pi_N^{\perp}(x) := \sum_{k=N+1}^{\infty} \langle x, e_k\rangle_2 e_k.$$

Let

$$\rho_3 = \frac{1}{2} \sup_{1\le k\le N+1} \mu_k^2.$$

We have

$$\|B(x)\|^2_{\mathscr{L}_2(L^2(\mu);H)} = \sum_{i=1}^{N+1} \|B(x)e_k\|^2_H = \sum_{k=1}^{N} \mu_k^2 \frac{\langle x, e_k\rangle_2^2}{\lambda_k} + \mu_{N+1}^2\|\pi_N^{\perp}(X_t)\|^2_H \le 2\rho_3\|x\|^2_H.$$

By (3.3) and (3.5), $(H1)$ holds for $\rho_1 = \theta_1, \rho_2 = \theta_2$ and

$$\mathrm{d}M_t = 2\langle B(X_t)\mathrm{d}W_t, X_t\rangle_H = 2\sum_{k=1}^{N} \frac{\mu_k}{\lambda_k}\langle X_t, e_k\rangle_2^2\langle \mathrm{d}W_t, e_k\rangle_2 + 2\mu_{N+1}\|\pi_N^{\perp}(X_t)\|^2_H\langle \mathrm{d}W_t, e_{N+1}\rangle.$$

Obviously,

$$\begin{aligned}
4\Big(\sum_{k=1}^{N+1} \mu_k^2\Big)\|X_t\|^4_H\mathrm{d}t \ge \mathrm{d}\langle M\rangle_t &= 4\Big\{\sum_{k=1}^{N} \frac{\mu_k^2}{\lambda_k^2}\langle X_t, e_k\rangle_2^4 + \mu_{N+1}^2\|\pi_N^{\perp}(X_t)\|^4_H\Big\}\mathrm{d}t \\
&\ge \frac{4\Big(\sum_{k=1}^{N} \frac{\langle X_t, e_k\rangle_2^2}{\lambda_k} + \|\pi_N^{\perp}(X_t)\|^2_H\Big)^2}{\sum_{k=1}^{N+1} \mu_k^{-2}}\mathrm{d}t = \frac{4\|X_t\|^4_H\mathrm{d}t}{\sum_{k=1}^{N+1} \mu_k^{-2}}.
\end{aligned}$$

Let

$$c_1 = \frac{4}{\sum_{k=1}^{N+1} \mu_k^{-2}}, \quad c_2 = 4\sum_{k=1}^{N+1} \mu_k^2.$$

Therefore, if $\mu_k^2 > 0$ for $1 \le k \le N+1$, then (2.2) holds for $g_i(s) = c_i s^2,\ i = 1, 2$.

Corollary 3.6. *Assume that (3.1) and (3.3) hold. Let* $\rho_3 = \frac{1}{2}\sup_{1\le k\le N+1}\mu_k^2$*. Then for any* $\theta \in \big(0, \frac{2(1+r)}{d(1-r)}\big) \cap (0,1]$*, all assertions in Theorems 2.1 hold for solutions to (3.4) with* B *given in (3.20). If moreover* $\mu_k^2 > 0$ *for* $1 \le k \le N+1$*, the assertion in Theorem 2.2 holds for* $g_i(s) = c_i s^2,\ i = 1, 2$.

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